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Minimal functional observer for linear time invariant systems

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Abstract—**This paper deals with the design of the minimal functional observer for linear time invariant systems. The proposed minimal functional observer estimates the linear function characterizing a state feedback controller law.**
A synthesis procedure of a minimum observer which has the same dimension as the control vector is then presented and a condition of such observer existence is stated.
The effectiveness and the availability of the minimal functional observer are highlighted through a numerical simulation to reconstruct the state feedback control law in order to stabilize the considered system.

***Keywords*— Linear system, minimal functional observer, state feedback controller.**

I. Introduction

In many real world engineering applications, the knowledge of the system state is often required not only for control purpose but also for monitoring and fault diagnosis. In practice, the measurements of the system state can be very difficult or even impossible, for example when an appropriate sensor is not available. Model-based state estimation is then a largely adopted strategy [1], [2], [3], [4], [5], [6] . Typically a state estimation is provided by means of an observer which inputs are composed by the inputs and the outputs of the system and outputs are the estimated states. Note that the structure of an observer is based on the mathematical model of the considered system. Therefore, the accuracy of the state estimation depends on the accuracy of the used mathematical model and the quality of the employed measurements.

Clearly, the state estimation is of fundamental importance with a wide range of application, like state feedback design, fault detection, digital image processing, and even cryptography. Especially, the use of a state observer (estimator) is necessary to realize a successful closed-loop control. For these reasons state estimator has gained constantly high interest in the literature [5], [6], [7], [8],[9], [10]. However in some applications it is sufficient to estimate directly linear functions of the state [11], [15],[16].

The design of observers which estimate linear functions of the state, as low-dimensional functional observers, were first studied in [11]. Later on, other approaches have been developed to design linear functional observers [11], [13], [14], [15], [16], [17].

Until now the direct design of a minimal observer of a given linear functional is an open question. Since the results developed in [11], several design schemes have been proposed to reduce order of the observer. These design methods are mainly based on the determination of matrices such that the Sylvester equation is fulfilled [18], [19]. Unfortunately, the problem rests in satisfying several conditions. Indeed, in many cases, necessary and sufficient existence conditions for the observer to be minimal are obtained for the fixed-pole observer problem only [20].

In this paper we consider the minimum functional observer for linear time invariant systems which estimates a linear functional characterizing a state feedback control law. In this way we present a procedure of synthesis of the minimum observer which has the same dimension as the control vector and we state a condition of such observer existence.

An outline of this paper is as follows: the studied minimal functional state observer and the proposed condition of its existence are presented in section II. Then, the section III is devoted to illustrate the proposed approach on a numerical example.

II. Minimal Functional State Observer

Consider the dynamical system defined by the linear state equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where:

- $t \in R^{+}$ is the continuous time,
- $x(t) \in R^{n}$ is the state vector,
- $u(t) \in R^{m}$ is the input vector,
- $y(t) \in R^{p}$ is the measurable output vector,
- A, B and C are constant matrices of appropriate dimensions.

The problem considered here is to design a minimal state observer to estimate the control law $u(t)$, such as:

$$u(t) = Lx(t) \quad (2)$$

where $L \in R^{m\times n}$ is a constant full rank $(m\times n)$ matrix.

The equations of a such observer are as follows :

$$\begin{cases} \dot{w}(t) = \hat{A}w(t) + \hat{B}u(t) + \Gamma y(t) \\ \hat{u}(t) = Ew(t) + Fy(t) \\ w(0) \text{ is arbitrary} \end{cases} \quad (3)$$

where $w(t) \in R^q$ is the observer state vector, $\hat{u}(t) \in R^m$ is the estimation of the function sate vector $u(t)$ (the feedback control law), $\hat{A} \in R^{q\times q}$, $\hat{B} \in R^{q\times m}$, $\Gamma \in R^{q\times p}$, $E \in R^{mxq}$ and $F \in R^{q\times p}$, are constant matrices of appropriate dimensions, such that $\hat{u}(t) \in R^q$ is an asymptotic estimate of the linear functions $u(t) = Lx(t)$ [20], [21], i.e.

$$\lim_{t\to\infty}[u(t) - \hat{u}(t)] = 0 \quad (4)$$

According to the value *q* of the observer state vector *w*(*t*), we distinguish several cases:

- $q = n$: the Luenberger observer;
- $q = n - p$: the reduced-order observer [22], [23];
- $m < q < n - p$ and q is such that no observer of the linear function $Lx(t)$ with an order less than q exists : the minimal-order observer or minimal observer [20];
- $q = m$: the minimum-order observer or minimum observer. *m* is the lower bound for the order of the observer (3) [24] , [25]

In this work we study the existence of a minimum functional observer of the control law $u(t) = Lx(t) \in R^m$ which means that the observer state vector $w(t)$ is of dimension *q*=*m*. Thus the matrix *E* in the observer equations (3) is taken the identity matrix $E = I_m$. Our purpose in the following development is to obtain the other matrix parameters of the observer and to find a sufficient condition of existence of a such observer.

Reconsider then the minimum observer equations (3) in which $E = I_m$. The estimation error between the control linear function (2) and its minimal state observer (3) is defined as :

$$e(t) = u(t) - \hat{u}(t) \quad (5)$$

Then, the estimation error dynamic equation is:

$$\dot{e}(t) = L\dot{x}(t) - \dot{\hat{u}}(t) \quad (6)$$

From (1) and (3) we have:

$$\begin{aligned} \dot{e}(t) &= L\left(Ax(t) + Bu(t)\right) - \dot{w}(t) - F\dot{y}(t) \\ &= \hat{A}\,e(t) + (LA - \hat{A}L - \Gamma C - FCA + \hat{A}FC)\,x(t) \\ &\quad + (LB - \hat{B} - FCB)\,u(t) \end{aligned} \quad (7)$$

The following theorem summarizes how the matrices $\hat{A}, \hat{B}$, Γ and F have to be chosen such that the estimation error dynamic (7) is asymptotically stable for all initial conditions $x(t_0)$ and $w(t_0)$.

Theorem 1: [13], [20], [26] The linear functional $\hat{u}(t)$ of (3) is an asymptotic estimate of the linear functional $Lx(t)$ of (2) for any $x(t_0)$, $w(t_0)$ and $u(t)$ if:

$$LA - \hat{A}L - \Gamma C - FCA + \hat{A}FC = 0 \quad (9)$$

$$\hat{B} = LB - FCB \quad (10)$$

where $\hat{A}$ is a Hurwitz matrix.

Note that, when the conditions of Theorem 1 are satisfied the estimation error dynamic (7) becomes $\dot{e}(t) = \hat{A}e(t)$ where $\hat{A}$ is a Hurwitz matrix i.e. the error *e*(*t*) tends to zero.

Minimum functional observer existence:
a dimension condition

The design of the functional observer (3) is reduced to the resolution of equations (9) and (10). As mentioned in [13], [14], the equation (9), which is a Sylvester one, is difficult to solve. In [27], necessary and sufficient conditions for the existence and stability of the functional observer (3) are given. The characterization of the minimum observer require to set the estimation error state matrix $\hat{A}$ and to solve the equation (9) in order to obtain the gain matrices $\Gamma(m\times p)$ and $F(\mathrm{m}\times p)$.

The matrix equation (9) is composed by $m\times n$ scalar linear equations of $2\times m\times p$ unknown parameters.

Notice that the matrix $\hat{A}$ must be taken such that the functional observer error converges asymptotically into zero faster than the system dynamics.

A dimension condition of existence of an exact solution (Γ, F) of equation (9) can be stated as follows:

$$2m\times p \geq m\times n$$

$$\text{i.e.} \quad 2p \geq n \quad (11)$$

III. NUMERICAL EXAMPLE

In this section we intend to illustrate the developments we made above for the synthesis of a minimum order observer.

Let us consider the fourth order, single input and two outputs system described by the following linear time invariant model:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -2 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) \end{cases}$$

with:

- $x(t) \in R^4$ the state vector ,
- $u(t) \in R$ the control vector,
- $y(t) \in R^2$ the output vector,
- $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -2 & -3 \end{pmatrix}$ the state matrix,
- $B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ the input matrix,
- $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ the output matrix of the system.

The task is to estimate the state linear functional $u(t) = Lx(t)$ where L is the feedback control gain, $L = [-15, -31, -22, -5]$ which has been chosen to place the system poles to the multiple value $\lambda = -2$. Note that the control law depends on all the state variables in particular the non measured ones $x_2(t)$ and $x_4(t)$.

This system has $n = 4$ state variables, $m = 1$ input variable and $p = 2$ output variables. Then, one can easily verify that the above stated dimension condition (11) is satisfied since we have $2p = n = 4$. Hence, a one-dimensional functional observer can be designed to estimate the feedback control law.

With the observer dynamic chosen as $\hat{A} = -5$, the matrix equation (9) leads to the following system of four algebraic equations which unknowns are the component of the matrices $\Gamma = [\Gamma_1 \quad \Gamma_2]$ and $F = [f_1 \quad f_2]$:

$$\begin{cases} -70.34 - \Gamma_1 - 5f_1 = 0 \\ -165.67 - f_1 = 0 \\ -131.4 - \Gamma_2 - 5f_2 = 0 \\ -32.08 - f_2 = 0 \end{cases}$$

These equations yield the following minimum observer gains:

$$\Gamma = [758.01 \quad 29]$$

$$F = [-165.67 \quad -32.08]$$

$$\hat{B} = -5.01$$

The performances of the proposed minimal functional observer are tested by numerical simulation.

The figure 1 shows the unstable behavior of the uncontrolled system. The figure 2 represents the evolution of all the state variables of the system provided with the observed control law (using the minimum functional observer).

The control signal evolution is shown in the figure 3 where the actual feedback control variable $(u(t) = Lx(t))$ and its estimation $\hat{u}(t)$ using the minimum functional observer are compared.

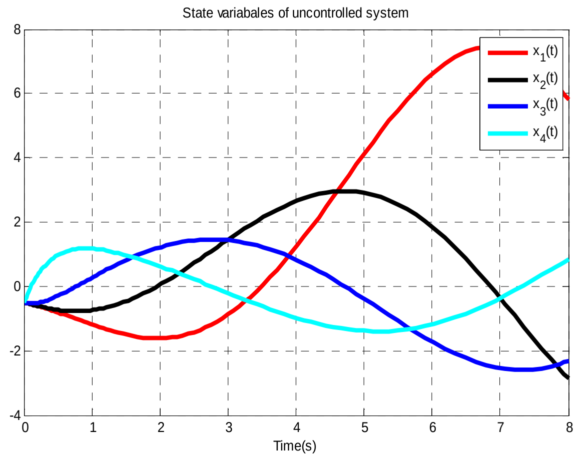

Fig.1 Evolution of the state variables of the considered unstable system

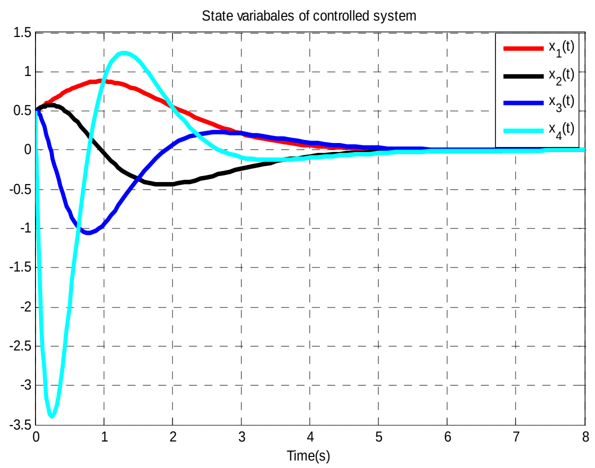

Fig.2 Evolution of the state variables of the system provided with the estimated state feedback controller law

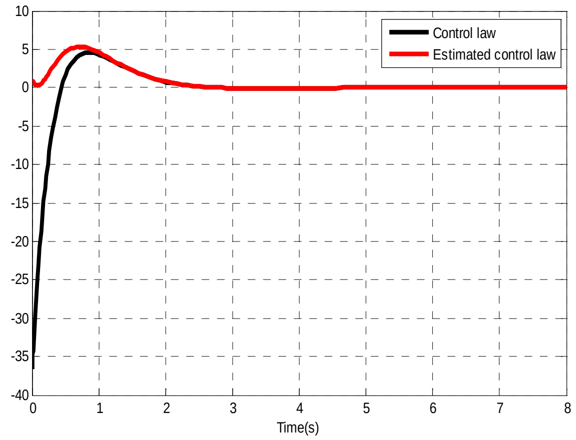

Fig.3 Evolution of the actual and estimated control law

III. Conclusion

The synthesis of a minimal functional observer for linear time invariant systems has been investigated. We have considered especially the observer design of a state linear function representing a feedback control law. In this way a dimension condition of a solution existence of the minimum order functional observer has been stated.

The implementation of the proposed development has been illustrated on a fourth order, single input double output linear system. The obtained results confirm the availability of the developed approach.

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